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MODELING THE PHASE INTERACTION IN THE MOMENTUM EQUATIONS OF A FLUID–SOLID MIXTURE†

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Abstract—The phase interaction term in the momentum equations for the flow of a fluid–solid mixture is derived using a control volume/control surface approach. This approach has been used by the authors in an earlier work to obtain the solid phase stress. An extension of this approach provides a consistent derivation for the necessary constitutive equations for a laminar flow of a dilute fluid–solid mixture. A fundamental difference is observed between the present model and other models in the literature. The diffusion/anti-diffusion phenomenon caused by the concentration gradient is discussed. The resulting governing equations are applied to the case of a vertical flow where the phase interaction term is first evaluated according to a previous model in the literature, and then to the one presently obtained. The resulting solutions show that a significant difference exists in the flow behavior described by the two solutions.

Key Words: two-phase, mixture, interaction, fluid–solid, particle, momentum, equation.

1. INTRODUCTION

Mass and momentum balance equations for a fluid–solid mixture flow have been given in the literature as follows (e.g. Ishii 1975):

$$\frac{\partial \rho^s}{\partial t} + \nabla \cdot (\rho^s \mathbf{u}) = 0, \qquad [1a]$$

$$\rho^s\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \rho^s \boldsymbol{g} + \mathbf{m} + \nabla \cdot (c\mathbf{T}^s); \qquad [1b]$$

and

$$\frac{\partial \rho^f}{\partial t} + \nabla \cdot (\rho^f \mathbf{v}) = 0, \qquad [1c]$$

$$\rho^f\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = \rho^f \boldsymbol{g} - \mathbf{m} + \nabla \cdot [(1-c)\mathbf{T}^f]. \qquad [1d]$$

In the above, $\rho^s = \rho_s c$ and $\rho^f = \rho_f(1-c)$ are the partial densities of the solid and fluid phase, respectively, where c is the solid concentration and ρ_s and ρ_f are the solid and fluid material density, respectively; $\mathbf{u}$ and $\mathbf{v}$ represent the solid and fluid velocities, respectively; the phase interaction force per unit volume of mixture is denoted by $\mathbf{m}$ and the stress for the solid and fluid phase are $\mathbf{T}^s$ and $\mathbf{T}^f$, respectively. In order to close the above equations, constitutive relations must be obtained for the phase interaction force and the phase stresses.

Many models in the existing literature have adopted the above equations for two-phase flow problems. However, when it comes to the constitutive relations for the solid phase stress $\mathbf{T}^s$ and the phase interaction $\mathbf{m}$, there are many distinctly different models. For instance, consider a general

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flow of a mixture of rigid particles and a Newtonian fluid. The phase stresses for such a flow have been modeled as

$$\mathbf{T}^{s} = -p^{s}\mathbf{I} + \mathbf{T}^{s\prime} \qquad [2a]$$

and

$$\mathbf{T}^{f} = -p^{f}\mathbf{I} + \mathbf{T}^{f\prime}, \qquad [2b]$$

where p^{s} and p^{f} are the phase pressures, $\mathbf{I}$ is the unit tensor and $\mathbf{T}^{s\prime}$ and $\mathbf{T}^{f\prime}$ are the deviatoric parts of the phase stresses. The solid phase pressure has been equated to the: (a) fluid phase pressure p^{f} (Drew 1976); (b) averaged fluid pressure around the surface of a particle (Drew 1983; Givler 1987); and (c) hydrostatic fluid pressure (Ahmadi 1987). The phase interaction term $\mathbf{m}$ has been modeled as

$$\mathbf{m} = n\mathbf{h} + \tilde{p}\nabla c, \qquad [3]$$

where n is the number of particles per unit volume, $\mathbf{h}$ is the hydrodynamic force per particle and $\tilde{p}$ has been equated to the: (a) solid phase pressure p^{s} (Ishii 1975; Drew 1983); (b) fluid phase pressure p^{f} (McTigue *et al.* 1986; Givler 1987); and (c) hydrostatic fluid pressure (Ahmadi 1987).

The above models do not differ because they describe different flow regimes. In fact, they are all intended for the most general flows of a fluid–solid mixture. This puzzling phenomenon can be attributed to the following reason. In the past, most two-phase flow models have been constructed mainly from principles of continuum mechanics. Mathematical consistency and physical plausibility are the requirements in this approach. All the above models have been considered to satisfy these requirements.

From a micromechanical point of view, however, the solid phase is a collection of discrete particles. Therefore, the dynamics of the solid phase should depend on many "particulate" parameters, such as the particle Reynolds number. The mathematical formulation of the relation between particulate parameters and the constitutive relations is beyond the scope of continuum mechanics. On the other hand, in the micromechanical approach, all terms in the governing and constitutive equations are obtained from information at the level of individual particles. The final form of these equations obviously must still satisfy the constraint imposed by continuum mechanics.

In an earlier work (Hwang & Shen 1989), the authors provided a derivation of the solid phase stress using a micromechanical approach. In that work, a control volume/control surface served as a convenient mental picture for constructing conservation laws. This same mental picture had been adopted previously by Prosperetti & Jones (1984) in their derivation of the momentum equations of a two-phase flow. Through this approach, the solid phase stress and the associate pressure term were first decomposed into collisional, kinetic and hydrodynamic parts. The hydrodynamic part was then derived based on the hydrodynamic force acting on individual particles. The result was identical to that of Batchelor (1970) and Batchelor & Green (1972), in which a volume averaging approach had beed used. This stress from hydrodynamic interaction was named the "particle-presence stress" following the terminology "stress due to the presence of particles" (Batchelor 1970).

The essence of this earlier work by the authors was to provide an interpretation of the solid phase stress based on fundamental physical laws that operate at the particulate level. Such an interpretation provides the bridge necessary to link micro and macro quantities. It is therefore an essential part for modeling the governing equations of a two-phase flow, in which the dynamic of macro quantities are described.

The present work is a continuation of the earlier work. Here, we extend the same approach utilized in Hwang & Shen (1989) to derive the constitutive relation for the phase interaction term $\mathbf{m}$. The momentum equations are completely described in terms of the forces that act on the individual particles. These momentum equations are the same as those obtained in Prosperetti & Jones (1984). The assumptions used in Prosperetti & Jones (1984) are not necessary in the following analysis. For a dilute laminar flow with low particle Reynolds number, all forces acting on the particles are known explicitly. A simplified vertical flow problem is then solved in this special regime. The solution of this problem is intended to demonstrate the importance of the micromechanical approach.

2. DERIVATION

Consider a mixture of a Newtonian fluid and rigid particles with uniform size. The fluid phase is assumed incompressible. The mass and momentum balance laws are given in [1a–d]. As discussed in Hwang & Shen (1989), the fluid and solid phase stress may be decomposed, respectively, as

$$\mathbf{T}^{\mathrm{f}} = \mathbf{T}^{\mathrm{v}} + \mathbf{T}^{\mathrm{t}} \qquad [4a]$$

and

$$\mathbf{T}^{\mathrm{s}} = \mathbf{T}^{\mathrm{c}} + \mathbf{T}^{\mathrm{k}} + \mathbf{T}^{\mathrm{p}}, \qquad [4b]$$

where $\mathbf{T}^{\mathrm{v}}$ is the fluid viscous stress, $\mathbf{T}^{\mathrm{r}}$ is the fluid turbulence stress (or Reynolds stress), $\mathbf{T}^{\mathrm{c}}$ is the collision/contact stress, $\mathbf{T}^{\mathrm{k}}$ is the kinetic stress (equivalent to the solid turbulence stress) and $\mathbf{T}^{\mathrm{p}}$ is the particle-presence stress resulting from the hydrodynamic forces acting on the particles. The physical interpretation of the fluid stress components is well-established. Similarly, the two components $\mathbf{T}^{\mathrm{c}}$ and $\mathbf{T}^{\mathrm{k}}$ in the solid stress have been derived in the literature of granular flows (e.g. Lun *et al.* 1984). The physical interpretation for $\mathbf{T}^{\mathrm{p}}$ was first given by Batchelor (1970) and Batchelor & Green (1972) using a volume averaging concept. The resulting stress has the following components:

$$T^{\mathrm{p}}_{ij} = \frac{1}{V_0}\left(\int_{A_0} \Sigma_{ik} n_k r_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V\right), \qquad [5]$$

where V_0 and A_0 are the volume and the surface of a single particle, Σ_{ik} is the hydrodynamically-induced local stress at $\mathrm{d}A$ on the surface of a particle or at $\mathrm{d}V$ inside a particle, n_k is the kth component of a unit outward normal on the particle's surface and r_j is the jth component of the position vector of the infinitesimals $\mathrm{d}A$ or $\mathrm{d}V$. Only one particle size V_0 is considered.

In Hwang & Shen (1989), a control volume/control surface approach is adopted to derive the particle-presence stress $\mathbf{T}^{\mathrm{p}}$. An identical result to that shown in [5] was obtained. This control volume/control surface approach is consistent with the concept used in deriving $\mathbf{T}^{\mathrm{c}}$ and $\mathbf{T}^{\mathrm{k}}$. In addition, it gives an alternative interpretation of $\mathbf{T}^{\mathrm{p}}$ that is physically transparent. In Hwang & Shen (1989), two important results have also been derived after obtaining [5]. First, it was shown that the last term in [5] is related to a single particle's rotation:

$$\frac{1}{V_0}\int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V = \tfrac{1}{5}\rho_{\mathrm{s}} R^2(\varepsilon_{ilj}\dot{\Omega}_l + \Omega_i\Omega_j - \Omega_k\Omega_k\delta_{ij}), \qquad [6]$$

where R is the particle's radius and $\boldsymbol{\Omega}$ is its angular velocity. In Babić (1989), this term has been shown to cancel with the rotational contribution in $\mathbf{T}^{\mathrm{k}}$. Second, the solid phase pressure is derived based on the definition used in continuum mechanics, where

$$p^{\mathrm{s}} = -\tfrac{1}{3}T^{\mathrm{s}}_{ii}, \qquad [7]$$

in which the contribution from the particle-presence stress is

$$p^{\mathrm{p}} = -\frac{R^3}{3V_0}\int_0^{2\pi}\int_0^{\pi}(-p\delta_{ik} - \tfrac{2}{3}\mu e_{ll}\delta_{ik} + 2\mu e_{ik})n_k n_i \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta. \qquad [8]$$

The above applies to cases where the flow around the surface of the particles is laminar, and μ is the absolute viscosity of the fluid. (In fact, if the flow is turbulent, [8] is still applicable if μ is identified as the "eddy" viscosity.) If the pressure and velocity field around a particle is known, the above equation may be integrated to give the particle-presence pressure for the solid phase explicitly. For a dilute incompressible flow of vanishing particle Reynolds number defined by $\mathrm{Re} = (\rho_{\mathrm{f}} RU)/\mu$, where U is the relative velocity between the particle and the surrounding fluid, the above reduces to

$$p^{\mathrm{p}} = \frac{R^3}{3V_0}\int_0^{2\pi}\int_0^{\pi} p \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta, \qquad [9]$$

which is identical to what Givler (1987) proposed through a heuristic argument. The basic expression given by [8] may be used to model other flow regimes when [9] no longer applies.

We now proceed to the modeling of the phase interaction term $\mathbf{m}$. The identical control volume/control surface approach used in deriving $\mathbf{T}^{\mathrm{p}}$ will be used here. All notations that follow represent local quantities in a single realization. Averaged quantities will be denoted explicitly.

Consider the control volume V shown in figure 1a. The surface of V denoted by S is the control surface of the mixture. Particles which sit entirely inside the control surface are called "inner particles", while particles cut by the control surface are called "surface-particles". The "solid portion" of the control surface, S_{e}, is defined by the intersection of the control surface and the surface particles. The remaining part of S is the "fluid portion" of the control surface. Forces acting on S_{e} account for the solid phase stress and those on $S - S_{\mathrm{e}}$ account for the fluid phase stress. There are also forces acting on the interface between the fluid and the solid within the control volume V. This interface is denoted by S_{i} in figure 1a and consists of the entire surface of all "inner particles" and the portion of the surface interior to S of all the "surface particles". The total phase interaction force inside V, $\int_V \mathbf{m}\,\mathrm{d}V$, is the total hydrodynamic force acting on S_{i}.

Let N be the number of inner particles in V and N' be the number of inner as well as surface particles of V. The total hydrodynamic force acting on S_{i} includes those that act on the whole particles, $\mathbf{h}_k, k = 1, \ldots, N$, and those that act on the partial surfaces. Therefore

$$\begin{aligned}\int_V \mathbf{m}\,\mathrm{d}V &= \int_{S_{\mathrm{i}}} \hat{\mathbf{n}} \cdot \mathbf{\Sigma}\,\mathrm{d}S_{\mathrm{i}} \\ &= \sum_{k=1}^{N} \mathbf{h}_k + \int_{A_{\mathrm{i}}} \hat{\mathbf{n}} \cdot \mathbf{\Sigma}\,\mathrm{d}A_{\mathrm{i}},\end{aligned} \qquad [10]$$

where A_{i} is the portion of the total surface particles' surface which is interior to the control volume; Σ is the local stress tensor on the particles due to the hydrodynamic effect of the surrounding fluid flow; and $\hat{\mathbf{n}}$ is the unit outward vector from the solid phase.

Let $\mathbf{h}^{\mathrm{p}}$ denote the integral of the surface force acting on any arbitrary volumetric portion V_{p} of a particle. In terms of the stress tensor $\mathbf{\Sigma}$ inside the particle induced by the hydrodynamic force on the surface of the particle, this resultant surface force may be expressed as

$$\begin{aligned}\mathbf{h}^{\mathrm{p}} &= \int_{S_{\mathrm{p}}} \hat{\mathbf{n}} \cdot \mathbf{\Sigma}\,\mathrm{d}S_{\mathrm{p}} \\ &= \int_{V_{\mathrm{p}}} \nabla \cdot \mathbf{\Sigma}\,\mathrm{d}V_{\mathrm{p}},\end{aligned} \qquad [11]$$

where S_{p} denotes the surface area of V_{p}. If one specifies V_{p} to be the outer or inner portion of a surface particle, as shown in figure 1b, the corresponding $\mathbf{h}^{\mathrm{p}}$ becomes $\mathbf{h}^{\mathrm{ou}}$ or $\mathbf{h}^{\mathrm{i}}$, respectively. As shown in [12], $\mathbf{h}^{\mathrm{i}}$ may be resolved into the forces on A_{i} and S_{e}, or

$$\mathbf{h}^{\mathrm{i}} = \mathbf{d}^{\mathrm{i}} + \mathbf{t}, \qquad [12]$$

as shown in figure 1b. Therefore

$$\int_{A_{\mathrm{i}}} \hat{\mathbf{n}} \cdot \mathbf{\Sigma}\,\mathrm{d}A_{\mathrm{i}} = \sum_{k=N+1}^{N'} \mathbf{d}_k^{\mathrm{i}} = \sum_{k=N+1}^{N'} (\mathbf{h}_k^{\mathrm{i}} - \mathbf{t}_k). \qquad [13]$$

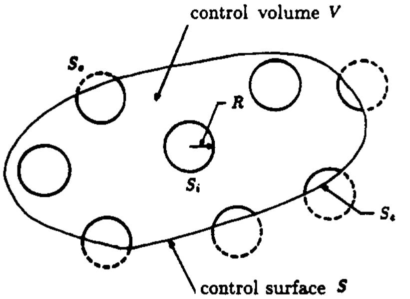

Figure 1a. Arbitrary control volume in a fluid–solid mixture flow.

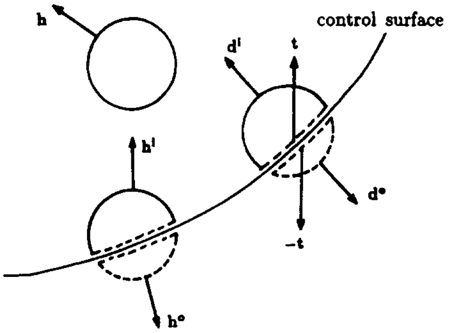

Figure 1b. Definition sketch of the surface forces acting on a surface particle.

By substituting [13] into [10],

$$\int_V \mathbf{m}\,\mathrm{d}V = \sum_{k=1}^{N} \mathbf{h}_k + \sum_{k=N+1}^{N'} \mathbf{h}'_k - \sum_{k=N+1}^{N'} \mathbf{t}_k. \qquad [14]$$

In integral notation, [14] becomes

$$\int_V \mathbf{m}\,\mathrm{d}V = \int_{V_s} \tilde{\mathbf{h}}\,\mathrm{d}V_s - \int_{S_e} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma}\,\mathrm{d}S_e, \qquad [15]$$

where V_s is the total volume of the solid phase interior to the control volume and $\tilde{\mathbf{h}} = \nabla \cdot \boldsymbol{\Sigma}$ is the total surface force per unit volume acting on a given infinitesimal volume $\mathrm{d}V$ in V_s. Equation [15] can be further reduced to

$$\int_V \mathbf{m}\,\mathrm{d}V = \int_V c\tilde{\mathbf{h}}\,\mathrm{d}V - \int_S \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} c\,\mathrm{d}S, \qquad [16]$$

where $c = 0$ or 1 depends on whether a fluid or solid phase are in the corresponding infinitesimals, respectively. Applying the ensemble average, $\langle\ \rangle$, to the above equation yields

$$\int_V \langle \mathbf{m} \rangle\,\mathrm{d}V = \int_V \langle c\tilde{\mathbf{h}} \rangle\,\mathrm{d}V - \int_S \hat{\mathbf{n}} \cdot \langle c\boldsymbol{\Sigma} \rangle\,\mathrm{d}S, \qquad [17]$$

Introducing the volume/mass-weighted average $\{\psi\} = \langle c\psi \rangle / \langle c \rangle$ to the above equation and applying Gauss's theorem to the second term on the right, one obtains

$$\int_V \langle \mathbf{m} \rangle\,\mathrm{d}V = \int_V \langle c \rangle \{\tilde{\mathbf{h}}\}\,\mathrm{d}V - \int_V \nabla \cdot (\langle c \rangle \{\boldsymbol{\Sigma}\})\,\mathrm{d}V. \qquad [18]$$

Because $\{\tilde{\mathbf{h}}\}$ represents the hydrodynamic force per unit volume in a particle, hence

$$\{\tilde{\mathbf{h}}\} = \frac{\{\mathbf{h}\}}{V_0}, \qquad [19]$$

where $\{\mathbf{h}\}$ is the mass-weighted average of the total hydrodynamic force acting on a particle that occupies the infinitesimal $\mathrm{d}V$ in space. The above is an approximation that applies when the length scale of ∇c is much larger than the particle's diameter, as shown in the appendix. This approximation is consistent with the derivation of the collisional and kinetic stresses, $\mathbf{T}^c$ and $\mathbf{T}^k$ respectively, in the granular flow literature, and the hydrodynamic solid phase stress, $\mathbf{T}^p$, in Hwang & Shen (1989). Substituting [19] into [18] and removing the integral sign yield

$$\langle \mathbf{m} \rangle = \frac{\langle c \rangle}{V_0} \{\mathbf{h}\} - \nabla \cdot (\langle c \rangle \{\boldsymbol{\Sigma}\}). \qquad [20]$$

In the above equations $\boldsymbol{\Sigma}$ represents the hydrodynamically-induced stress on the surface S_e. This surface, as previously mentioned, is the area of intersection between the surface particles and the control surface. Therefore, the term $\{\boldsymbol{\Sigma}\}$ is exactly the particle-presence stress $\mathbf{T}^p$ defined in [4b]. At this point all variables are averaged quantities, one may remove the $\langle\ \rangle$ and $\{\ \}$ in [20] to yield a simpler expression:

$$\mathbf{m} = \frac{c}{V_0} \mathbf{h} - \nabla \cdot (c\mathbf{T}^p), \qquad [21]$$

where $c/V_0 = n$ is the number of particles per unit volume. Equation [21] applies to flows of a general fluid–solid mixture. The only restriction is that particles are uniform in size and shape. This restriction may be removed with little additional work.

The hydrodynamic force, $\mathbf{h}$, acting on a single particle is, in general, more complicated than the drag force $\mathbf{f}$. For a dilute mixture, $\mathbf{h}$ is approximated by hydrodynamic forces on a single particle in an infinite fluid flow. With the additional assumption of a vanishing particle Reynolds number, Maxey & Riley (1983) have rigorously derived such forces for a general flow field. Based on their work, Hwang (1989) has obtained a simpler expression as follows:

$$\mathbf{h} = \mathbf{f}_S + \mathbf{f}_a + V_0 \nabla \cdot \mathbf{T}^f, \qquad [22]$$

where $\mathbf{f}_S$ is the Stokes drag acting on a particle; and $\mathbf{f}_a$ stands for the additional forces including the added mass effect, the Basset force (Basset 1888) and the Saffman force (Saffman 1965) due to the fluid inertia.

To the authors' knowledge, a solution does not exist yet for the hydrodynamic force acting on a particle for the case of a general transient flow with finite solid concentration. An explicit solution for $\mathbf{h}$, given in [22], therefore awaits future development in fluid mechanics.

In conclusion, the phase interaction term obtained from the present model is given in [21]. The result applies to a general flow of a fluid–solid mixture with uniform particle size. In the case of a dilute mixture with low particle Reynolds number, the present model gives

$$\mathbf{m} = \frac{c}{V_0}(\mathbf{f}_S + \mathbf{f}_a) + c\nabla \cdot \mathbf{T}^f - \nabla \cdot (c\mathbf{T}^p). \qquad [23]$$

3. DISCUSSIONS ON THE GOVERNING EQUATIONS

A fundamental difference is observed when comparing [3], where $\mathbf{m} = n\mathbf{h} + \tilde{p}\nabla c$, with [21], where $\mathbf{m} = n\mathbf{h} - \nabla \cdot (c\mathbf{T}^p)$. However, since $\mathbf{m}$ is only one of the terms in the momentum equations, one naturally suspects that through different bookkeeping for the momentum balance, perhaps the individual terms such as $\mathbf{m}$ and $\mathbf{T}^s$ are different, but the resulting equations are the same. After all, the control volume/control surface is one of the many ways that may be used to construct conservation laws. In order to see whether the bookkeeping might be the problem, we compare the resulting momentum equations with another set of equations obtained previously.

Substituting [4b] and [21] into [1b], the solid phase momentum equation is obtained as

$$\rho^s\left(\frac{\partial\{\mathbf{u}\}}{\partial t} + \{\mathbf{u}\} \cdot \nabla\{\mathbf{u}\}\right) = \rho^s \mathbf{g} + \nabla \cdot [c(\mathbf{T}^c + \mathbf{T}^k)] + \frac{c}{V_0}\mathbf{h}, \qquad [24]$$

where $\mathbf{T}^k = -\rho_s\{\mathbf{u}''\mathbf{u}''\}$ and $\mathbf{u}'' = \mathbf{u} - \{\mathbf{u}\}$ is the fluctuation velocity of a particle. Substituting [4a] and [21] into [1d], the fluid phase momentum equation is obtained as

$$\rho^f\left(\frac{\partial\{\mathbf{v}\}}{\partial t} + \{\mathbf{v}\} \cdot \nabla\{\mathbf{v}\}\right) = \rho^f \mathbf{g} - \frac{c}{V_o}\mathbf{h} + \nabla \cdot (c\mathbf{T}^p) + (1-c)\nabla \cdot \mathbf{T}^f - \mathbf{T}^f \nabla c, \qquad [25]$$

where $\mathbf{T}^f = \mathbf{T}^v - \rho_f\{\mathbf{v}''\mathbf{v}''\}$ is the total fluid stress including both the viscous stress and the Reynolds stress. In the latter, $\mathbf{v}'' = \mathbf{v} - \{\mathbf{v}\}$ is the turbulence velocity of the fluid phase. For a dilute laminar flow with vanishing particle Reynolds number, we substitute [22] into [24] and [25] and drop $\mathbf{T}^c$ and $\mathbf{T}^k$ to obtain

$$\rho^s\left(\frac{\partial\{\mathbf{u}\}}{\partial t} + \{\mathbf{u}\} \cdot \nabla\{\mathbf{u}\}\right) = \rho^s \mathbf{g} + \frac{c}{V_0}(\mathbf{f}_S + \mathbf{f}_a) - c\nabla p^f + c\nabla \cdot \mathbf{T}^{f'} \qquad [26]$$

and

$$\begin{aligned}\rho^f\left(\frac{\partial\{\mathbf{v}\}}{\partial t} + \{\mathbf{v}\} \cdot \nabla\{\mathbf{v}\}\right) = {} & \rho^f \mathbf{g} - \frac{c}{V_0}(\mathbf{f}_S + \mathbf{f}_a) + (1-c)(-\nabla p^f + \nabla \cdot \mathbf{T}^{f'}) \\ & + \nabla \cdot c(\mathbf{T}^{p'} - \mathbf{T}^{f'}) - \nabla c(p^p - p^f). \qquad [27]\end{aligned}$$

In the above, $\mathbf{T}^{f'} = \mu[\{\nabla\mathbf{v}\} + \{\nabla\mathbf{v}\}^T]$ is the deviatoric part of the fluid stress and $\mathbf{T}^{p'}$ is the deviatoric part of the particle-presence stress.

Equations [26] and [27] are different from those obtained in the previous literature. [Except that from the work of Prosperetti & Jones (1984), in which these two equations can also be deduced.] For instance, the momentum equations in McTigue *et al.* (1986) are

$$\rho^s\left(\frac{\partial\{\mathbf{u}\}}{\partial t} + \{\mathbf{u}\} \cdot \nabla\{\mathbf{u}\}\right) = \rho^s \mathbf{g} + \frac{c}{V_0}(\mathbf{f}_S + \mathbf{f}_a) - c\nabla p^p - (p^p - p^f)\nabla c \qquad [28]$$

for the solid phase and

$$\rho^f\left(\frac{\partial\{\mathbf{v}\}}{\partial t} + \{\mathbf{v}\} \cdot \nabla\{\mathbf{v}\}\right) = \rho^f \mathbf{g} - \frac{c}{V_0}(\mathbf{f}_S + \mathbf{f}_a) + \nabla \cdot [2\mu(1 + \tfrac{5}{2}c)\mathbf{D}_f] - \nabla p^f + c\nabla p^f \qquad [29]$$

for the fluid phase. In which, $\mathbf{D}_f$ is the strain-rate tensor defined by $\frac{1}{2}[\nabla\{\mathbf{v}\} + (\nabla\{\mathbf{v}\})^T]$.

There are a number of differences between the models described by [26] and [27] and the model described by [28] and [29]. One of the main contrasts between the two models is the term $-(p^{p}-p^{f})\nabla c$. In the present model, this term appears in the fluid momentum equation, [27], as part of the last term on the right. In the model given by McTigue *et al.* (1986), this term appears in the solid momentum equation [28]. Depending on the relative value of the solid and fluid phase pressure, this term can be positive, negative or zero. The mathematical significance of this term was first discussed by Stuhmiller (1977). Numerical instability can occur if phase pressures are equal. Givler (1987) discussed this term from a physical point of view.

When the solid phase pressure is different from the fluid phase pressure, the term $-(p^{p}-p^{f})\nabla c$ will produce an equivalent force in the momentum equation where this term appears. According to McTigue *et al.* (1986), as well as Givler (1987), this term is in the solid momentum equation. In the case of an inviscid flow around the particles, $p^{p}-p^{f}=-\frac{1}{4}\rho_{f}(\mathbf{v}-\mathbf{u})^{2}$, the direction of this force is therefore identical to ∇c. This "anti-diffusive" phenomenon was considered in Givler (1987) as non-physical.

In the present model, there is no $-(p^{p}-p^{f})\nabla c$ term in the solid momentum equation. Hence the difference in the phase pressure $p^{p}-p^{f}$ at first glance will not force the solid to move in or opposite to the direction of ∇c. However, the term $-(p^{p}-p^{f})\nabla c$ appears in the fluid equation, hence it will force the fluid phase to move in or opposite to the direction of ∇c. In the case of an inviscid flow, $p^{p}-p^{f}=-\frac{1}{4}\rho_{f}(\mathbf{v}-\mathbf{u})^{2}$, the fluid phase will then move in the direction of ∇c. In the case of a Stokes' flow, $p^{p}-p^{f}=\frac{9}{32}\rho_{f}(\mathbf{v}-\mathbf{u})^{2}$, the fluid phase will move opposite to ∇c. A direct physical explanation for this phenomenon follows.

Consider a dilute flow of a Newtonian fluid with spherical particles. The phase velocities are different, but both the fluid and solid phases are moving uniformly. Initially, the fluid flow is steady and uniform without a velocity gradient and any other inertia effects, therefore, terms such as $\mathbf{f}_{a}$, $\mathbf{T}^{f'}$ and $\mathbf{D}_{f}$ are not present, and ∇p^{f} is in the flow direction. As shown in figure 2, assume that a small perturbation of the uniform particle concentration is introduced, so that the concentration in region A is greater than that in region B. If, later, the particles are forced to move from region A to region B, diffusion results. The opposite direction corresponds to anti-diffusion. In this given flow condition, the only possible lateral forces in the fluid–solid momentum equations are created by $\mathbf{f}_{S}$ and ∇c. Initially, $\mathbf{f}_{S}$ is in the flow direction also. The ∇c term is present as soon as the perturbation is introduced.

Because the flow is dilute, the hydrodynamic interaction of the fluid and solid phase is closely approximated by the superposition of individual particles in an infinite fluid. Consider first that the flow is in the Stokes regime. The fluid pressure at the surface of the spherical particle is (Chester & Breach 1969)

$$\frac{p-p_{\infty}}{\dfrac{\mu U}{R}}=-\frac{3}{2}\left(1+\frac{3}{8}\mathrm{Re}\right)\cos\theta+\frac{27}{32}\mathrm{Re}\cos^{2}\theta+O(\mathrm{Re}^{2}\log\mathrm{Re}), \qquad [30]$$

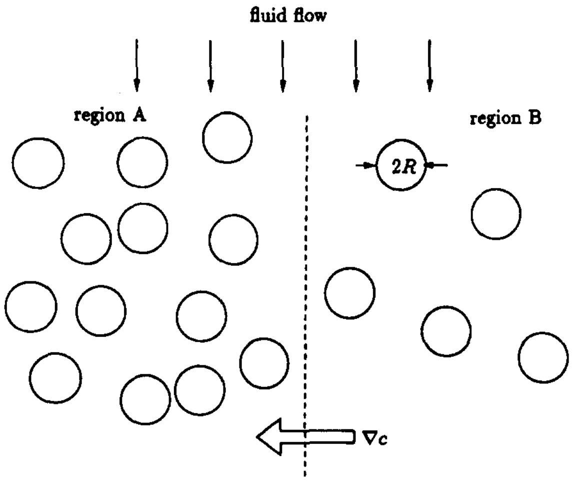

Figure 2. A uniform fluid–solid flow with a concentration gradient.

where p_∞ is the undisturbed fluid pressure, U and Re are defined above [9] and the angle θ is measured from the direction of the relative velocity. On the basis of a single particle's analysis, the average fluid pressure at the surface of the particle is increased from p_∞ by

$$\frac{\int_0^{2\pi}\int_0^{\pi}(p-p_\infty)R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\int_0^{2\pi}\int_0^{\pi}R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}=\frac{9}{32}\mathrm{Re}\frac{\mu U}{R}+O(\mathrm{Re}^2\log\mathrm{Re}). \qquad [31]$$

The tendency to increase the fluid pressure will only diminish to zero at infinite distance. Since each particle increases the surrounding fluid pressure, superposition implies that the fluid pressure will increase more as the particle concentration increases. Therefore, fluid will flow from the high particle concentration region to regions of low particle concentration. Namely, from region A to region B in figure 2 or opposite to the direction of ∇c. Similarly, if the flow is in the inviscid regime, by applying the potential flow theory and Bernoulli's equation, the fluid pressure surrounding a particle is reduced from the undisturbed pressure. The higher the particle concentration, the greater is this reduction. Therefore, fluid is driven from the higher fluid pressure zone, where the particle concentration is low, to the low fluid pressure zone, where the particle concentration is high, or in the direction of ∇c. Therefore, the diffusion/anti-diffusion takes place in the fluid phase. The particles will also move. However, their motion will be produced by $\mathbf{f}_\mathrm{S}$ and will move in the direction of the fluid's motion. If the above argument is correct, it means that perturbation of concentration will induce secondary flows in the concentration gradient direction.

The above arguments are based on the basic principles of fluid mechanics for a uniformly flowing, infinite fluid surrounding a single particle. Its conclusion is applicable to extremely dilute systems. To experimentally verify the diffusion/anti-diffusion phenomena is not easy because in most experimental setups, the presence of boundaries will create additional forces from the flow gradient and curvature effect (such as the Saffman's force). The presence of other particles also affect the applicability of the above results. The clever design of an experiment that will test the ∇c effect alone is not obvious.

4. A SPECIAL CASE

The discussion on diffusion/anti-diffusion shows that a fundamental difference exists between the present model and a previous model. In this section we study another special case, which is intended to demonstrate that a qualitatively different solution may be obtained for a fluid–solid flow if the phase interaction term $\mathbf{m}$ is modeled differently from that developed in this investigation. To show this, a boundary value problem is solved using two different models: model A, based on [3]; and model B, based on [21]. In [3], $\tilde{p}$ is equated to p^s following Ishii's (1975) model.

A Poiseuille flow of the mixture through a vertical two-dimensional channel will be studied in this section. Since the major interest here is to compare the two different models of $\mathbf{m}$ as described by [3] and [21], all other terms in the governing equations and constitutive equations for $\mathbf{T}^\mathrm{s}$, are kept the same in the comparison.

For a Poiseuille flow the continuity equation provides no information for it is automatically satisfied. Only momentum equations need to be considered. Model A, which corresponds to [3], can be expressed as follows, where x is parallel to the channel and gravity direction, and y is perpendicular to the channel.

Momentum equation for the solid phase:
x-component,

$$0=-\rho_\mathrm{s}cg+\frac{9\mu c}{2R^2}(v_x-u_x)-c\frac{\partial p_\mathrm{s}}{\partial x}+\frac{5\mu}{2}\left(\frac{\partial v_x}{\partial y}\frac{\partial c}{\partial y}+c\frac{\partial^2 v_x}{\partial y^2}\right); \qquad [32a]$$

and
y-component,

$$0=-c\frac{\partial p_\mathrm{s}}{\partial y}. \qquad [32b]$$

Momentum equation for the fluid phase:
x-component,

$$0 = -\rho_f(1-c)g - \frac{9\mu c}{2R^2}(v_x - u_x) - (1-c)\frac{\partial p_f}{\partial x} + \mu\left[\frac{\partial v_x}{\partial y}\frac{\partial(1-c)}{\partial y} + (1-c)\frac{\partial^2 v_x}{\partial y^2}\right]; \qquad [32c]$$

and
y-component,

$$0 = -p_s\frac{\partial c}{\partial y} + p_f\frac{\partial c}{\partial y} - (1-c)\frac{\partial p_f}{\partial y}. \qquad [32d]$$

where g is the gravitational acceleration and v_x and u_x are the velocity components in the x-direction for the fluid phase and solid phase, respectively.

Similarly, equations for model B based on the present modeling of **m** given in [21] can be expressed as follows.

Momentum equation for the solid phase:
x-component,

$$0 = -\rho_s c g + \frac{9\mu c}{2R^2}(v_x - u_x) - c\frac{\partial p_f}{\partial x} + c\mu\frac{\partial^2 v_x}{\partial y^2}; \qquad [33a]$$

and
y-component,

$$0 = -c\frac{\partial p_f}{\partial y}. \qquad [33b]$$

Momentum equation for the fluid phase:
x-component,

$$\begin{aligned} 0 = & -\rho_f(1-c)g - \frac{9\mu c}{2R^2}(v_x - u_x) + c\frac{\partial p_f}{\partial x} - c\mu\frac{\partial^2 v_x}{\partial y^2} - c\frac{\partial p_s}{\partial x} \\ & + \frac{5\mu}{2}\left(\frac{\partial v_x}{\partial y}\frac{\partial c}{\partial y} + c\frac{\partial^2 v_x}{\partial y^2}\right) - (1-c)\frac{\partial p_f}{\partial x} \\ & + \mu\left[\frac{\partial v_x}{\partial y}\frac{\partial(1-c)}{\partial y} + (1-c)\frac{\partial^2 v_x}{\partial y^2}\right]; \end{aligned} \qquad [33c]$$

and
y-component,

$$0 = c\frac{\partial p_f}{\partial y} - c\frac{\partial p_s}{\partial y} - p_s\frac{\partial c}{\partial y} - p_f\frac{\partial(1-c)}{\partial y} - (1-c)\frac{\partial p_f}{\partial y}. \qquad [33d]$$

One can prove easily that the solid volume fraction c is constant with respect to y for both models. In addition, $\partial p_s/\partial x$ is equal to $\partial p_f/\partial x$ in this problem. The boundary conditions associated with the above equations are $v_x = 0$ at the wall and $\partial v_x/\partial y = 0$ at the centerline. The solid phase velocity u_s has an algebraic relation with v_s for both models.

Using the following non-dimensional variables,

$$y^* = \frac{y}{w}, \quad \rho_s^* = \frac{\rho_s}{\rho_f}, \quad \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^* = -\frac{\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)}{(\rho_f g)}, \quad v_x^* = \frac{v_x}{\left(\frac{w^2}{\mu}\rho_f g\right)} \quad \text{and} \quad u_x^* = \frac{u_x}{\left(\frac{w^2}{\mu}\rho_f g\right)},$$

where w is the width of the two-dimensional channel, the above equations may be solved analytically.

The solution from model A gives

$$v_x^* = \frac{1}{2\left(1+\frac{3c}{2}\right)}\left[\rho_s^* c + (1-c) - \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^*\right] y^{*2} - \frac{1}{8\left(1+\frac{3c}{2}\right)}\left[\rho_s^* c + (1-c) - \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^*\right]$$

and

$$u_x^* = v_x^* + \frac{2(1-c)}{9\left(1+\frac{3c}{2}\right)}\left(\frac{R}{w}\right)^2\left[\frac{5}{2} - \rho_s^* - \frac{3}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^*\right].$$

The fluid velocity v_x^* from model B is identical to that from model A. The solution for u_x^* of model B is

$$u_x^* = v_x^* + \frac{2}{9\left(1+\frac{3c}{2}\right)}\left(\frac{R}{w}\right)^2\left[-\left(1+\frac{c}{2}\right)\rho_s^* + \frac{3c}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^* + (1-c)\right].$$

Comparisons between models A and B can be deduced from the above solutions. As shown in figure 3, the effect of ρ_s^* on $u_x^* - v_x^*$ is the same for both models. Namely, as ρ_s^* increases from the buoyant case to the settling case, $u_x^* - v_x^*$ decreases. The pressure gradient, however, has a totally different influence on the two models. With an increasing pressure gradient, the phase velocity difference increases slightly according to model B, while it decreases rapidly according to model A. Consequently, in model A, settling particles can move ahead of the fluid and buoyant particles can lag behind the fluid. In model B, settling particles lag behind the fluid (unless the pressure gradient approaches infinity) and the buoyant particles always move ahead of the fluid. This highlights the qualitative difference between the two models that results solely from the interaction term **m**.

The above solution is a simplified version of a real vertical fluid–solid flow. Because the hydrodynamic force on the particles is approximated by the drag force only, near-wall phenomena cannot be accurately modeled. This example only serves as a demonstration. Namely, to show that constitutive relations significantly impact upon the qualitative behavior of the governing equations.

5. CONCLUSIONS

In an earlier work by the authors, the control volume/control surface approach was introduced to the flow of a fluid–solid mixture to formulate the solid phase stress. This approach is extended in the present study to obtain the phase interaction in the momentum equations. A consistent procedure is thus developed to derive the governing equations for a fluid–solid flow in terms of micromechanical information.

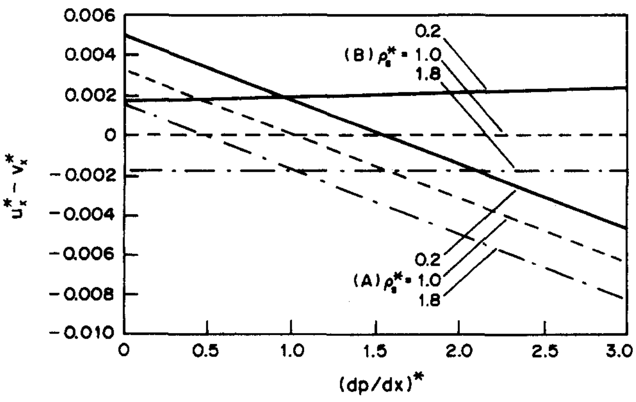

Figure 3. Effect of density ratio and pressure gradient on the phase velocity difference.

When the mixture is flowing in a laminar regime and when the mixture is dilute with vanishing particle Reynolds number, the system of equations are completely modeled. The resulting equations are different from many existing models.

One significant difference in the present governing equations from some of those derived previously is the diffusion/anti-diffusion phenomenon discussed first by Givler (1987). The ∇c term that occurs due to the phase pressure difference appears in the fluid momentum equation instead of the solid momentum equation. From a physical explanation, it is believed that the trend predicted by the present model is reasonable. However, experimental verification is desirable.

The derived governing equations are applied to solve a simplified laminar vertical flow problem. Two different phase interaction models are used to test their impact on the flow behavior. A qualitative difference in the solution is observed. This emphasizes the importance of a consistent procedure, that utilizes micromechanics, to derive every term in the governing equations.

In this work, only the momentum equations are considered. The same approach should be extended to the derivation of the terms in the energy equations. Such an extension is necessary when the particle's inertia becomes important and/or the fluid becomes turbulent. We plan to discuss this in a subsequent article.

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APPENDIX

In [19] it is given that the average hydrodynamic force per unit volume at a point in space is equal to the average total hydrodynamic force per particle occupying that point divided by the particle's volume,

$$\{\tilde{\mathbf{h}}\} = \frac{\{\mathbf{h}\}}{V_0}. \qquad [19]$$

This is an approximation that applies when the length scale of the concentration gradient is much larger than the particle's diameter. The proof is given below.

Consider a point P in space, as shown in figure A1. In n realizations when P is occupied by a particle, the hydrodynamic force acting on these n particles are represented by $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n$. By definition,

$$\begin{aligned}
\{\mathbf{h}\} &= \frac{1}{n}\sum_{i=1}^{n} \mathbf{h}_i \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\int_{V_0} \tilde{\mathbf{h}}\,\mathrm{d}V\right)_i \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{V_0} \rho_s[\dot{\mathbf{u}}_i^c - \boldsymbol{g} + (\omega_i \times \mathbf{r})]\,\mathrm{d}V \\
&= \frac{1}{n}\sum_{i=1}^{n}\rho_s\left[V_0(\dot{\mathbf{u}}_i^c - \boldsymbol{g}) + \dot{\omega}_i \times \int_{V_0}\mathbf{r}\,\mathrm{d}V + \omega_i \times \left(\omega_i \times \int_{V_0}\mathbf{r}\,\mathrm{d}V\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\rho_s V_0(\dot{\mathbf{u}}_i^c - \boldsymbol{g}), \qquad [\mathrm{A.1}]
\end{aligned}$$

where $\mathbf{u}_i^c$ is the velocity of the center of the ith particle and ω_i is its rigid body rotation velocity. In the above, $\int_{V_0}\mathbf{r}\,\mathrm{d}V = 0$ has been utilized. The equation of motion for an infinitesimal volume $\mathrm{d}V$ at P occupied by a particle is $\nabla \cdot \boldsymbol{\Sigma} = \rho_s(\dot{\mathbf{u}} - \boldsymbol{g}) = \rho_s[\dot{\mathbf{u}}^c - \boldsymbol{g} + (\omega \times \mathbf{r})]$, where obviously $\nabla \cdot \boldsymbol{\Sigma}$ is evaluated at P and $\mathbf{u}^c$ is the velocity of the center of a particle. Hence, the above can be further reduced to

$$\begin{aligned}
\{\mathbf{h}\} &= \frac{V_0}{n}\sum_{i=1}^{n}[(\nabla \cdot \boldsymbol{\Sigma})_i - \rho_s(\omega_i \times \mathbf{r})] \\
&= V_0\{\nabla \cdot \boldsymbol{\Sigma}\} - \rho_s V_0\{\omega \times \mathbf{r}\} \\
&= V_0\{\nabla \cdot \boldsymbol{\Sigma}\} - \rho_s V_0(\{\dot{\omega} \times \mathbf{r}\} - \{\mathbf{r} \cdot \omega\omega\} + \{\omega \cdot \omega\mathbf{r}\}) \\
&= V_0\{\nabla \cdot \boldsymbol{\Sigma}\} - \rho_s V_0(\{\dot{\omega}\} \times \{\mathbf{r}\} - \{\mathbf{r}\} \cdot \{\omega\omega\} + \{\omega \cdot \omega\}\{\mathbf{r}\}). \qquad [\mathrm{A.2}]
\end{aligned}$$

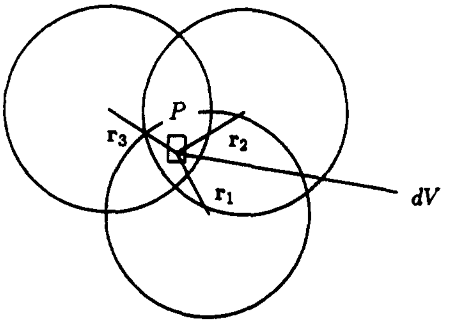

Figure A1. An infinitesimal volume at a point in space.

In the above, the fact that no correlation between the position vector $\mathbf{r}$ and the rigid particle rotation ω has been applied.

The mass-weighted position vector $\{\mathbf{r}\}$ is the ensemble average of all vectors $\mathbf{r}_i$ from the center of a particle that occupies $\mathrm{d}V$ in the ith realization to that point at $\mathrm{d}V$ (figure A1). If there is concentration gradient,

$$\{\mathbf{r}\} = \{\tilde{\mathbf{r}} P(\tilde{\mathbf{r}})\}, \qquad [A.3]$$

where $\tilde{\mathbf{r}}$ is a uniformly distributed position vector and $P(\tilde{\mathbf{r}})$ is the probability density function of having $\tilde{\mathbf{r}}$ in $\mathrm{d}V$. Expanding $P(\tilde{\mathbf{r}})$ into Taylor series and truncating at the first order,

$$P(\tilde{\mathbf{r}}) = 1 - \tilde{\mathbf{r}} \cdot \frac{\nabla c}{c}. \qquad [A.4]$$

Utilizing the following non-dimensionalization:

$$\tilde{\mathbf{r}}' = \frac{\tilde{\mathbf{r}}}{R}, \quad c' = \frac{c}{c_{\mathrm{ave}}}, \quad \nabla' c' = \frac{\nabla c}{\frac{c_{\mathrm{ave}}}{L}},$$

where L is the characteristic length of the concentration gradient, we have

$$P(\tilde{\mathbf{r}}) = 1 - \frac{R}{L} \tilde{\mathbf{r}}' \cdot \frac{\nabla' c'}{c'}. \qquad [A.5]$$

Since all non-dimensional terms are of the same order, $P(\tilde{\mathbf{r}}) \rightarrow 1$ as $R/L \rightarrow 0$ and, therefore, $\{\mathbf{r}\} = 0$ as $R/L \rightarrow 0$. Thus, from [A.2], $\{\nabla \cdot \Sigma\} = \{\tilde{\mathbf{h}}\} = \{\mathbf{h}\}/V_0$, as given in [19].

It is worth noting that in the current granular flow literature, the derivation of the collisional stress, $\mathbf{T}^{\mathrm{c}}$, and the kinetic stress, $\mathbf{T}^{\mathrm{k}}$, also adopted the same approximation. Namely, the concentration gradient is assumed to have a length scale much larger than the particle's diameter. This assumption is used when letting the radial distribution of the neighboring particles be an isotropic function of the local concentration at a point only. Furthermore, in deriving the solid phase stress (Hwang & Shen 1989), the particle's location on a control surface has been assumed as equally distributed along the diameter of a particle. This is equivalent to the assumption of a large length scale of the concentration grandient. The current derivation of the phase interaction term is therefore consistent with the other constitutive relations in the governing equations. The application of these constitutive relations are thus restricted to the case where the concentration gradient has a much larger length scale than the particle's diameter.